Math2050A Term1 2017
Tutorial 1, Sept 14

Exercises

1. Let $S := (a, b]$, where $a < b$. Find $\inf S$, $\sup S$.
2. Let $S := \{\frac{n}{2^n} : n \in \mathbb{N}\}$. Find $\inf S$, $\sup S$.
3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence s.t. $a_n \geq 0$. Define

$$\sum_{n=1}^{\infty} a_n := \sup\{a_1,\, a_1 + a_2,\, a_1 + a_2 + a_3, ...\}$$

Suppose $f : \mathbb{N} \to \mathbb{N}$ is a bijective map. Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{f(n)}$. By definition, LHS is $\sup\{a_{f(1)},\, a_{f(1)} + a_{f(2)},\, a_{f(1)} + a_{f(2)} + a_{f(3)} ...\}$

4. Let $\{a_{ij}\}_{i,j\in\mathbb{N}}$ with $a_{ij} \geq 0$ for all i, j. Show that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$$

Comments

Exercise 3 is saying that the order of summation is not important when $a_n \geq 0$. This matters when there are infinitely many negative terms and positive terms in the sequence. You may google " Riemann's Rearrangement Theorem" for more information.

Another observation from Exercise 3: Given $\{a_i\}_{i\in I}$ with $a_i \geq 0$. Here I can be arbitrary set, probably uncountable. One can still define

$$\sum_{i\in I} a_i := \sup\{\sum_{i\in F} a_i : F \subset I,\, F \text{ is a finite set } \}$$

However, in this case, the supremum exists in $\mathbb{R}$ only when $a_i = 0$ except countably many i. You may try to prove this.

From the definition above, show that $\sum_{i,j\in\mathbb{N}} a_{ij} = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$ in the setting of exercise 4. It shall be more direct to show that all three terms in exercise 4 equal $\sum_{i,j\in\mathbb{N}} a_{ij}$. Nonetheless, see

Solution for exercise 4 only.
Define $s_{mn} := \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$. We will show that

(i)
$$\sup_m \sup_n s_{mn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

(ii)
$$\sup_n \sup_m s_{mn} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

(iii)
$$\sup_{m,n} s_{mn} = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$$

Before showing these, by our textbook [Bartle] p.46 Q12, **Principle of the Iterated Suprema**, we note that

$$\sup_m \sup_n s_{mn} = \sup_n \sup_m s_{mn} = \sup_{m,n} s_{mn}$$

Hence, it remains to show (i),(ii),(iii):
(i) implies (ii) by defining $b_{ij} := a_{ji}$ and $s'_{mn} := \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} = s_{nm}$.
For (iii), you may try it yourself. It is similiar to exercise 3.
For (i), we claim first: $\sup_m \sup_n s_{mn} \geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. Prove claim:
Fix any $m_0, n_0 \in \mathbb{N}$, note: $\sup_m \sup_n s_{mn} \geq \sup_n s_{m_0 n} \geq s_{m_0 n_0} = \sum_{i=1}^{m_0}\sum_{j=1}^{n_0} a_{ij}$.
Try to argue that $\sup_m \sup_n s_{mn} \geq \sum_{i=1}^{m_0}\sum_{j=1}^{\infty} a_{ij}$. Suppose not, there is $\epsilon > 0$ s.t.

$$\begin{aligned}\sup_m \sup_n s_{mn} &< \sum_{i=1}^{m_0}\sum_{j=1}^{\infty} a_{ij} - \epsilon \\ &= \sum_{i=1}^{m_0}(\sum_{j=1}^{\infty} a_{ij} - \frac{\epsilon}{m_0}) \\ &< \sum_{i=1}^{m_0}\sum_{j=1}^{N_i} a_{ij} \text{ for some } N_i \text{ (depending on } i)\end{aligned}$$

Now let $n_0 := \max\{N_1, ..., N_{m_0}\}$, we have $\sup_m \sup_n s_{mn} < s_{m_0 n_0}$, contradiction arises. Therefore, $\sup_m \sup_n s_{mn} \geq \sum_{i=1}^{m_0} \sum_{j=1}^{\infty} a_{ij}$ and the claim will then follow by definition of supremum.
The second claim is that $\sup_m \sup_n s_{mn} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$, which is easy.